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Solution of the Implicitly Discretised Fluid Flow Equations by Operator-Splitting

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A non-iterative method for handling the coupling of the implicitly discretised time-dependent fluid flow equations is described. The method is based on the use of pressure and velocity as dependent variables and is hence applicable to both the compressible and incompressible versions of the transport equations. The main feature of the technique is the splitting of the solution process into a series of steps whereby operations on pressure are decoupled from those on velocity at each step, with the split sets of equations being amenable to solution by standard techniques. At each time-step, the procedure yields solutions which approximate the exact solution of the difference equations. The accuracy of this splitting procedure is assessed for a linearised form of the discretised equations, and the analysis indicates that the solution yielded by it differs from the exact solution of the difference equations by terms proportional to the powers of the time-step size. By virtue of this, it is possible to dispense with iteration, thus resulting in an efficient implicit scheme while retaining simplicity of implementation relative to contemporary block simultaneous methods. This is verified in a companion paper which presents results of computations carried out using the method.

INTRODUCTION

Implicit finite-difference schemes for solving the equations governing the flow of fluids are frequently (and are fast becoming more so) favoured over their explicit counterparts. This is because of the unconditional stability of the former as contrasted with the stability of explicit methods which is subject to severe restrictions on the size of the time-step that can be taken. Such restrictions can drastically impair the efficiency of the algorithm particularly when applied to the calculation of steady-state flows, as is amply demonstrated in, for example, [1]. Moreover, explicit techniques can still be at a disadvantage even for time-dependent computations, because the time-step size necessary for procuring the required temporal accuracy is still likely to be significantly larger than that dictated by the explicit stability condition (especially if high-order schemes are used). This is because the stability limit is imposed by the local conditions in high-velocity regions of the flow regardless of the significance of the temporal variations prevailing there. Whereas the temporal variations whose accurate resolution is desired often prevail elsewhere

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in the domain where conditions allow taking longer time-steps without impairing accuracy.

If the method is to be general (e.g., equally applicable to both two and three dimensions), the primitive variables, i.e., density, pressure, and velocities, should be retained in the equations as the working variables. Of these, the density is often chosen to stand as a main dependent variable, wherein the pressure is evaluated from it via an equation of state. This approach is facilitated by the explicit appearance of the density in the continuity equation which may be regarded as one whose main dependent variable is density. Clearly, as the linkage between pressure and density weakens in the low Mach number range, the technique will falter, eventually breaking down in the incompressible flow limit. In this extreme, the variations of density cease to relate to those of pressure, thus causing the change in the rôle of the continuity equation to that of a compatibility condition for the velocity field. The many existing methods developed specifically for incompressible flows, for example, [2–5], surmount this problem by treating the pressure as a main dependent variable. This choice is in fact equally valid for compressible flows, which endows methods based on it with great versatility, as in, for example, [6, 7]. Such versatility is often required in the computation of low- and medium-speed subsonic flows, particularly when the flow attains different speeds either in different regions or at different instants (e.g., the flow in a reciprocating engine at different parts of its cycle).

In order to determine the pressure, which, while appearing in each of the momentum equations, vanishes from the continuity relation in the incompressible limit, a pressure equation is usually derived by joint manipulation of these equations (as in, for example, [6–9]). The resulting pressure equation replaces the continuity relation while the momentum equations retain their rôle for determining the velocity field, the equation set being coupled via the pressure and velocities.

Existing methods which solve the pressure–velocity coupled system fall into two categories, namely, semi- and fully implicit schemes. In the first group, the momentum equations are discretised in an explicit manner with the exception of the pressure gradient terms, which are treated implicitly; the continuity relation is also enforced implicitly (see, for example, [2, 3]). As a consequence, the equation-coupling reduces to a one-way linkage in which the pressure equation contains no time-advanced velocities and is hence solvable directly. Such schemes, however, because of their reliance on explicit differences, suffer from the already mentioned time-step restrictions. In the second category, the equations are discretised fully implicitly, with the coupling being handled through the use of iteration. Some methods which fall within this group employ sequential iteration (such as the SIMPLE method in [5] and SIMPLER in [8]), in which the equations for each variable are solved repeatedly in succession. Others utilise block iteration (such as the SIVA scheme in [9]), in which the variable block is solved simultaneously for a point (or a line) at a time. The advantage gained by implicit differencing of the equations is thus offset by the use of iteration which makes time-dependent calculations rather expensive as iteration is effected at each time-step. Moreover, the iterative process can be troublesome as convergence is not always assured and usually requires ad hoc specification of under-relaxation.

In this paper, a non-iterative method for handling the pressure–velocity coupling of the implicitly discretised fluid flow equations is presented. The method (called PISO for *P*ressure-*I*mplicit with *S*plitting of *O*perators) utilises the splitting of operations in the solution of the discretised momentum and pressure equations such that the fields obtained at each time-step are close approximations of the exact solution of the difference equations with a formal order of accuracy of the order of powers of δt depending on the number of operation-splittings used (two such splittings are proposed herein). Although this order of accuracy does not necessarily imply that the actual errors in the solution are insignificant, nor that the overall scheme employing the splitting procedure will possess the same order of accuracy, what is important is the fact that these errors decay rapidly as δt is refined, and do so as, or more, quickly than the errors embodied in the discretisation scheme itself. This, together with the finding that the stability of the overall scheme is little impaired by the splitting procedure (now depending on the choice of spatial discretisation scheme), allows getting rid of iteration while retaining the advantage of implicit differencing, namely, the ability to cope with large time-steps. This is verified in a companion paper [10], where the methodology is applied to the computation of both compressible and incompressible flow.

Although the method is cast in a time-dependent form, it is also useful for steady-state calculations due to its stability for fairly large time-steps. For such flow, the method in its incompressible version shares some features with existing iterative schemes, namely, SIMPLER in [8] and PUP in [14]. The main common feature is the use of a number of successive updates for the pressure and velocity fields at each pass (a time-step for PISO, an iteration for SIMPLER and PUP), in the endeavour to satisfy continuity and momentum simultaneously. The similarities and differences between the methods will become apparent when the present procedure is outlined. For transient flow, however, the other methods rely on iteration; whether they can be extended for implementation in a non-iterative, time-marching manner like PISO is a matter for conjecture. Furthermore, it will transpire later that the compressible version of PISO has much less in common with these iterative methods.

In the present paper, the PISO methodology is outlined and its accuracy assessed; some aspects of its stability are also examined. For the sake of convenience and ease of analysis, this is carried out with reference to the incompressible, isothermal flow equations. Also for the sake of clarity, the discretised equations used in the analysis are those which incorporate the Euler implicit temporal difference scheme, although in principle the method should be equally valid for other implicit schemes. The extension of the method to the fully compressible flow equations is then presented in the latter part of the paper. Finally, the predictor–corrector methodology is generalised to the treatment of the coupling between other transport equations, in particular that arising through the source terms in the equations for the $k-\varepsilon$ turbulence model.

EQUATIONS AND DISCRETISATION

The Transport Equations

The governing transport equations in Cartesian tensor notation, for momentum and continuity, are:

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_j u_i)= -\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\sigma_{ij}+S_i \tag{1}$$

and

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)=0 \tag{2}$$

where, in Eq. (1), σ_{ij} is the stress tensor which is related in some way to the velocity field, e.g., by a Newtonian constitutive law or possibly by an appropriate turbulence model, should such an approach be chosen to stimulate the effects of turbulence (see [11]). The term S_i contains the external source of momentum.

The pressure and the density are related by an equation of state which may, in general, be written in the form

$$\rho = p\phi(p, T) \tag{3}$$

where T is the temperature; for a perfect gas, ϕ becomes $1/RT$. Furthermore, the temperature is related to the total energy, e, which obeys its own transport equation, which is

$$\frac{\partial}{\partial t}(\rho e)+\frac{\partial}{\partial x_j}(\rho u_j e)=\frac{\partial \dot{q}_j}{\partial x_j}-\frac{\partial}{\partial x_j}(pu_j)+\frac{\partial}{\partial x_j}(u_i\sigma_{ij})+Q. \tag{4}$$

In Eq. (4), $\dot{q}_j$ is the diffusive flux of energy which is related in some way to e, and Q contains the external source of energy.

At low Mach numbers, the density becomes weakly related to the pressure and Eq. (3) ceases to apply. The governing equations (1) and (2) then reduce to their incompressible form, which for istothermal flow can be written as

$$\rho\frac{\partial u_i}{\partial t}+\rho\frac{\partial}{\partial x_j}(u_j u_i)= -\frac{\partial p}{\partial x_i}+\frac{\partial \sigma_{ij}}{\partial x_j}+S_i \tag{5}$$

and

$$\frac{\partial u_i}{\partial x_i}=0 \tag{6}$$

respectively.

The pressure equation may be derived at this stage from the differential form of the momentum and continuity equations. This practise, however, has its drawbacks, in that when the equations are discretised, it is not easy to ensure that all the terms in the pressure equation (which have their origin in the momentum and continuity equations) are discretised in a manner consistent with the discretisation of the corresponding terms in the parent equations. As a consequence of this inconsistency, the pressure obtained thus may not always yield a velocity field which satisfies both the momentum and continuity equations simultaneously. Since satisfaction of the latter is a requisite outcome of the solution of the pressure equation,[1] the aforementioned incompatibility becomes unacceptable.

Alternatively, the pressure equation may be derived from the discretised form of the continuity and momentum equations. This automatically ensures that the pressure equation is discretised in consistence with the momentum and continuity equations. It is this practice which is adopted presently, and is dealt with later.

Finite-Difference Formulation

The transport equations stated above may now be expressed in finite-difference form. There are numerous ways of representing the spatial and temporal derivatives in the parent equations by discrete analogues. However, it is not the task here to make a choice among the alternative differencing schemes, with the exception of the choice of implicit differencing, as is implied by the theme of the paper. Rather, the thrust of the study is to deal with the method of solving the difference equations, and in order to facilitate easy and clear presentation of this, the discretised equations are formulated here using the Euler implicit difference scheme. It should be emphasised, however, that the method of solution itself, and the related analysis presented herein, is not restricted to the use of that particular scheme.

Also for the sake of convenience, a symbolic operator form, which caters for most of the widely used spatial difference formulae, is employed here in presenting the discretised equations. Thus, if n and $n+1$ denote successive time levels, then for compressible fluids the governing equations (1), (2), and (4) may be expressed in difference form for each mesh point as

$$\frac{1}{\delta t}\{(\rho u_i)^{n+1}-(\rho u_i)^n\}=H(u_i^{n+1})-\Delta_i p^{n+1}+S_i \tag{7}$$

$$\frac{1}{\delta t}(\rho^{n+1}-\rho^n)+\Delta_i(\rho u_i)^{n+1}=0 \tag{8}$$

and

$$\frac{1}{\delta t}\{(\rho e)^{n+1}-(\rho e)^n\}=G(e^{n+1})-\Delta_i(p u_i)^{n+1}+J(u_i^{n+1})+Q \tag{9}$$

[1] Satisfaction of the continuity relation may still be achieved in this case by iteration on the pressure and momentum equations—a process which the present paper sets out to eliminate.

respectively. In Eq. (7), the operator H stands for the finite-difference representation of the spatial convective and diffusive fluxes of momentum, and the operator Δ_i, which also appears in Eq. (8) and (9), is the finite-difference equivalent of $\partial/\partial x_i$. The operator G in Eq. (9) stands for the discretised convective and diffusive fluxes of energy, while operator J contains the stress work terms. Operators H and G, in general, take the form

$$H(u_i) = A_m u_{i,m} \tag{10}$$

and

$$G(e) = B_m e_m \tag{11}$$

where suffix m is a grid node identifier and the summation is over all the nodes involved in the formulation of the finite-difference representation of the spatial fluxes. This formulation also determines the expressions for the A and B coefficients, which are functions of the velocities, densities, etc. Operators H and G are, therefore, non-linear and to enable the solution of Eq. (7) to (9), linearisation of the operators may be adopted. Alternatively, since a splitting procedure is to be effected to the various terms in the equations, the very same splitting can also be applied to the coefficients A and B in expressions (10) and (11). For the purpose of the present study, and in order to make the analysis tractable, the A and B coefficients are assumed to be constant over the interval δt, thus effectively linearising operators H and G.

The pressure equation may now be derived by taking the divergence of Eq. (7) and substituting for $\Delta_i\,(\rho u_i)^{n+1}$ in Eq. (8) to get

$$\frac{1}{\delta t}(\rho^{n+1} - \rho^n) + \Delta_i(\rho u_i)^n + \delta t\{\Delta_i H(u_i^{n+1}) - \Delta_i^2 p^{n+1} + \Delta_i S_i\} = 0 \tag{12}$$

which, when rearranged, becomes

$$\Delta_i^2 p^{n+1} = \Delta_i H(u_i^{n+1}) + \frac{1}{\delta t}\Delta_i(\rho u_i)^n + \Delta_i S_i + \frac{1}{\delta t^2}(\rho^{n+1} - \rho^n). \tag{13}$$

Equation (13), which replaces Eq. (8), is Poisson type in which Δ_i^2 is the Laplacian operator in difference form. This equation serves for determining p^{n+1}, while Eq. (7) retains its role for the velocity field u_i^{n+1}. Clearly, Eq. (7) and (13) are linearly coupled through the appearance of p^{n+1} and u_i^{n+1} in both; it is this coupling which is addressed in the present paper.

Similarly, for incompressible fluids, the discretised analogues of the transport equations (5) and (6) may be expressed in the same operator notation used above as

$$\frac{\rho}{\delta t}(u_i^{n+1} - u_i^n) = H(u_i^{n+1}) - \Delta_i p^{n+1} + S_i \tag{14}$$

and

$$\Delta_i u_i^{n+1} = 0. \tag{15}$$

The corresponding pressure equation which is derived by taking the divergence of Eq. (14) and substituting Eq. (15) is

$$\Delta_i^2 p^{n+1} = \Delta_i H(u_i^{n+1}) + \Delta_i S_i + \frac{\rho}{\delta t} \Delta_i u_i^n. \tag{16}$$

Equations (14) and (16) are those which respectively determine the velocity and pressure fields in incompressible flow.

Methodology for Incompressible Flow

Preamble

For ease of presentation, the proposed split-operator scheme (PISO) is first applied to the incompressible flow equations; this also renders the accuracy analysis of the procedure more tractable. Extension of the method to the compressible equations is made later in the paper.

The operative equations for the incompressible flow case are Eq. (14) to (16), which are coupled through the appearance of the time-advanced u_i and p in them. The present paper is preoccupied with handling this coupling of the equations rather than with methods of solving the simultaneous scalar sets of equations for the individual variables. It is, therefore, assumed that suitable standard methods (e. g., direct, iterative, or ADI) may be employed for the solution of equations like (14) and (16) when these are decoupled in variables as effected by PISO.

Operator-Splitting

The splitting of operators (or factorisation) is not a new concept and is often invoked either in temporal differencing as in [1] or in the solution of the discretised equations as in the ADI technique [12]. The principle is here extended to apply to the coupling between variables, namely, the pressure and velocity, whereby operations involving different variables are split into a series of predictor–corrector steps.

Let the superscripts *, **, and *** denote intermediate field values obtained during the splitting process. The equations can hence be factorised as follows:

(a) *Predictor Step.* The pressure field prevailing at t^n is used in the solution of the implicit momentum equations (14) to yield the u_i^* velocity field. Thus

$$\frac{\rho}{\delta t}(u_i^* - u_i^n) = H(u_i^*) - \Delta_i p^n + S_i. \tag{17}$$

Equation (17) is solvable by one of several standard techniques to yield the u_i^* field which, it should be noted, will not in general satisfy the zero-divergence condition (15).

(b) *First Corrector Step*. A new velocity field, u_i^{**}, together with a corresponding new pressure field, p^*, are now sought such that the zero-divergence condition

$$\Delta_i u_i^{**} = 0 \tag{18}$$

is met. For this, the momentum equation (14) is taken as

$$\frac{\rho}{\delta t}(u_i^{**} - u_i^n) = H(u_i^*) - \Delta_i p^* + S_i \tag{19}$$

which, it should be noted, is of explicit type in that $H(u_i)$ is taken to operate on the u_i^* field for reasons which will become apparent shortly. Equations (18) and (19) are used to derive the pressure equation

$$\Delta_i^2 p^* = \Delta_i H(u_i^*) + \Delta_i S_i + \frac{\rho}{\delta t}\Delta_i u_i^n \tag{20}$$

which is readily solvable since the right-hand side contains terms in the known field u_i^*, and this is the consequence of using the explicit form of Eq. (19). The p^* field obtained by solving Eq. (20) may be inserted into Eq. (19) to yield the u_i^{**} field, which, it should be recalled, satisfies the zero-divergence condition (18).

(c) *Second Corrector Step*. A new velocity field, u_i^{***}, together with its corresponding new pressure field, p^{**}, are formulated, such that

$$\Delta_i u_i^{***} = 0. \tag{21}$$

The operative momentum equation is now taken as the explicit-type equation

$$\frac{\rho}{\delta t}(u_i^{***} - u_i^n) = H(u_i^{**}) - \Delta_i p^{**} + S_i. \tag{22}$$

The corresponding pressure equation is therefore

$$\Delta_i^2 p^{**} = \Delta_i H(u_i^{**}) + \Delta_i S_i + \frac{\rho}{\delta t}\Delta_i u_i^n. \tag{23}$$

From Eq. (23), the p^{**} field can be readily determined since the right-hand side of the equation is known, and with this new pressure, the u_i^{***} field can be evaluated from Eq. (22).

More corrector steps such as the above can obviously be introduced. However, as will be shown later, the accuracy with which u_i^{***} and p^{**} approximate the exact solution u_i^{n+1} and p^{n+1} of Eq. (14) and (16) is sufficient for most practical purposes, which makes further corrector steps superfluous. On the other hand, it will also be shown that a minimum of two corrector steps must be taken before the velocity and pressure fields thus obtained can be legitimately regarded as solutions to Eq. (14) and (16).

Accuracy and Stability Considerations

The methodology presented above yields pressures and velocities which are only approximations of the exact solution. The task now is to analyse how accurate these approximations are. This is possible only in relation to a linearised form of the finite-difference equations. The findings in what follows should therefore be treated merely as guidelines indicating the likely performance of the method which has to be ultimately tested in actual computations. The heuristic analysis carried out (much of which is not presented here for the sake of brevity) is indeed intended for this purpose rather than to furnish general proofs, which are unlikely to apply to the actual non-linear equations. The analysis includes two approaches, the first of which is based on a Taylor series analysis. This provides an indication as to how the errors introduced in the splitting might decay with δt (with the spatial mesh size kept finite). Also presented is a separate examination of the influence of a non-vanishing velocity-divergence field on the pressure solution.

Let ε_i and ξ be the errors in approximating the velocity u_i and pressure p at any stage in the splitting. Hence, define

$$\varepsilon_i^k = u_i^{n+1} - u_i^k \tag{24}$$

where k stands for any one of the superscripts *, **, or ***. Similary, define

$$\xi^l = p^{n+1} - p^l \tag{25}$$

where l stands for any one of the superscripts n, *, or **. If u_i^{n+1} and p^{n+1} are to be approximated by u_i^{***} and p^{**}, the error introduced will then be ε_i^{***} and ξ^{**}, respectively, which ideally should be of the same order or smaller than the discretisation[2] errors introduced in the formulation of the finite-difference representation of the parent equations.

To begin with, Eq. (17) of the predictor stage is subtracted from the original momentum equation (14) to give

$$\frac{\rho}{\delta t}\varepsilon_i^* = H(\varepsilon_i^*) - \Delta_i \xi^n \tag{26}$$

[2] Discretisation errors should be distinguished here from truncation errors which are the errors in the representation of the individual terms in the differential equations (see [13]). In general, discretisation errors are one order higher than truncation errors.

where, because H is linear, it has been possible to replace the term $H(u_i^{n+1}) - H(u_i^*)$ by $H(\varepsilon_i^*)$. Now

$$\zeta^n \equiv p^{n+1} - p^n$$

which becomes, when p is expanded by Taylor series in t

$$\zeta^n \equiv p^n + O(\delta t) - p^n = O(\delta t)$$

where $O(\delta t)$ indicates the order of truncation errors. It then follows from Eq. (26) that

$$\varepsilon_i^* = O(\delta t^2).$$

The equations governing the first corrector step are (19) and (20) which, when subtracted from the original equations (14) and (16), give

$$\frac{\rho}{\delta t}\varepsilon_i^{**} = H(\varepsilon_i^*) - \Delta_i \zeta^* \tag{27}$$

and

$$\Delta_i^2 \zeta^* = \Delta_i H(\varepsilon_i^*). \tag{28}$$

Since it has just been found that ε_i^* is of the order δt^2, Eq. (28) dictates that ζ^* is also of $O(\delta t^2)$, while Eq. (27) now gives ε_i^{**} as $O(\delta t^3)$.

Similarly, the second corrector equations (22) and (23), subtracted from Eq. (14) and (16), are

$$\frac{\rho}{\delta t}\varepsilon_i^{***} = H(\varepsilon_i^{**}) - \Delta_i \zeta^{**} \tag{29}$$

and

$$\Delta_i^2 \zeta^{**} = \Delta_i H(\varepsilon_i^{**}). \tag{30}$$

Equation (30) reveals that since ε_i^{**} is $O(\delta t^3)$, then ζ^{**} is also $O(\delta t^3)$. Consequently, Eq. (29) leads to the result that ε_i^{***} is $O(\delta t^4)$. Thus, it has been shown that the approximations of the exact u_i^{n+1} and p^{n+1} fields produced by the two-corrector stage splitting are good to $O(\delta t^4)$ and $O(\delta t^3)$, respectively. Evidently, if more corrector stages are introduced, the order of accuracy is increased by one for each additional stage. However, since the original discretised equations themselves, i.e., (14) and (16), contain discretisation errors (of $O(\delta t^3)$ for second-order-accurate schemes and of $O(\delta t^2)$ or first-order ones), the endeavour to achieve higher accuracy in the solution of these equations appears to be unnecessary.

So far, the accuracy of the splitting technique has been examined in terms of a Taylor-series-based order of accuracy analysis. This analysis, however, is insufficient in that it is not always indicative of the absolute magnitudes of the errors involved. In what follows, there is a closer examination of the origins of errors arising in the determination of the pressure field.

Consider the pressure equation (16) in the absence of momentum sources and when the velocity field at t^n is divergence free; this is done here on the grounds of convenience and should not in any way invalidate the findings. Thus

$$\Delta_i^2 p^{n+1} = \Delta_i H(u_i^{n+1}). \tag{31}$$

The operator H given by expression (10) is a linear combination of the u_i's prevailing at each node in question and its neighbours. Insertion of expression (10) into Eq. (31) gives

$$\Delta_i^2 p^{n+1} = \Delta_i(A_m u_{i,m}^{n+1}). \tag{32}$$

The right-hand side of Eq. (32) can be decomposed[3] into

$$\Delta_i(A_m u_{i,m}) = \bar{A}_m \, \Delta_i u_{i,m} + \bar{u}_{i,m} \, \Delta_i A_m \tag{33}$$

where the bars on A and u_i denote suitable averages over the cluster of nodes involved in the spatial difference scheme defining expression (10). Since continuity demands the vanishing of the divergence of the velocity field everywhere, the $\Delta_i u_i$ term in expression (33) vanishes, resulting in

$$\Delta_i^2 p^{n+1} = \bar{u}_{i,m}^{n+1} \, \Delta_i A_m. \tag{34}$$

Thus, the pressure field can be seen to be generated by gradients in A_m which are the coefficients in the finite-difference representation of the convective and diffusive fluxes for which H stands.

Consider now the corresponding pressure equation of the first corrector step, i.e., Eq. (20), in the absence of sources. With the aid of relation (33), the equation can be written as

$$\Delta_i^2 p^* = \bar{u}_{i,m}^* \, \Delta_i A_m + \bar{A}_m \, \Delta_i u_{i,m}^* \tag{35}$$

where the term $\Delta_i u_i^*$ is the divergence of the predictor step velocity field u_i^* which, in general, does not vanish. Evidently, this is an error term as comparison of this equation with the exact one (34) will reveal. This implies that the p^* field can be dominated by mass errors generated in the intermediate stages of the calculation, errors which are often larger than the weak gradients of A that are responsible for setting up the pressure field. Hence, although p^* is apparently a second-order-

[3] This is the finite-difference equivalent to

$$\frac{\partial}{\partial x_i}(A u_i) = A \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial A}{\partial x_i}.$$

accurate approximation of p^{n+1}, Eq. (35) shows that it may still be a very poor solution.

Consider, on the other hand, the pressure equation (23) arising in the second corrector stage. With the aid of relation (33), it can be rewritten as

$$\Delta_i^2 p^{**} = \bar{u}_{i.m}^{**} \Delta_i A_m + \bar{A}_m \Delta_i u_{i.m}^{**}$$

which, because the continuity equation (18) is satisfied, reduces to

$$\Delta_i^2 p^{**} = \bar{u}_{i.m}^{**} \Delta_i A_m. \tag{36}$$

It is apparent that Eq. (36) is free from the mass errors which predominate in Eq. (35), and that the p^{**} field is generated solely by gradients in A as it should be. It is for this reason that at least two corrector stages must be implemented in order to obtain accurate pressures.

As for stability, the finite-difference scheme defined by Eq. (14) and (16) is fully implicit; the system would therefore be unconditionally stable if these equations were to be solved exactly. However, as the solution to the equations is achieved approximately, the residual errors in these approximations may alter the stability characteristics of the overall scheme. No attempt will be made here to rigorously analyse the stability of the method as this is dependent in the first instance on the particular spatial difference scheme used in arriving at the discretised equations. In any event, the stability of the overall method will be greatly affected by the non-linearity of the actual system of equations solved which is bound to impose its own restrictions on the time-step size. Nevertheless, a simple stability analysis such as the one that follows serves to illuminate some pitfalls and ways of circumventing them.

It is postulated here that if the product of the error amplification factors (in a von Neumann stability analysis) associated with each of the predictor and corrector steps is less than unity, then the overall linearised scheme is unconditionally stable. Inspection of the predictor step equation (17) reveals that it is implicit in u_i^* and always has an error amplification factor of less than unity. The corrector step equations (19) and (22), however, are explicit-like in which u_i may be construed as being advanced in a pseudo-time corresponding to the starred operations, wherein, the convective and diffusive fluxes (lumped into H) are evaluated at some intermediate pseudo-time (star) level. These equations can therefore have error amplification factors, defined as $\varepsilon_i^{**}/\varepsilon_i^*$ and $\varepsilon_i^{***}/\varepsilon_i^{**}$ and obeying Eq. (27) and (29), of greater than unity. It is easy to show that for a simple one-dimensional linearised equation in which either a first-order upwind or a centred spatial difference scheme is used, the amplification factors for Eq. (27) and (29) are less than unity only if the time-step size is smaller than a certain value. This δt threshold is comparable to the corresponding values relating to an explicitly differenced momentum equation. Thus, for time-steps larger than this threshold (which is the reason for using implicit methods in the first instance), the above splitting leads to departure from the exact solution of the difference equations over the time-step.

Although this may not undermine the stability of the overall scheme (except for very large δt), it can result in significant deterioration in accuracy, as was indeed found in practice.

To remedy the situation, the corrector stage equations are replaced by ones which are intrinsically more stable (i.e., with smaller amplification factors). The simplest way of doing so without losing either versatility or accuracy is to separate the central (i.e., diagonal) element of the operator H and shift it to the left-hand side of the corrector stage equations where it is treated implicitly, while the rest of the elements are retained on the right-hand side where they are still treated explicitly. Thus, if the central element in expression (10) is $A_0 u_i$, then define H' as

$$H'(u_i) = H(u_i) - A_0 u_i \tag{37}$$

where A_0 is the central coefficient which for most practical spatial difference schemes takes a finite negative value. Equations (19) and (20) of the first and second corrector steps are therefore replaced by

$$\left(\frac{\rho}{\delta t} - A_0\right) u_i^{**} - \frac{\rho}{\delta t} u_i^n = H'(u_i^*) - \Delta_i p^* + S_i \tag{38}$$

and

$$\left(\frac{\rho}{\delta t} - A_0\right) u_i^{***} - \frac{\rho}{\delta t} u_i^n = H'(u_i^{**}) - \Delta_i p^{**} + S_i \tag{39}$$

respectively.

An examination of the error propagation properties of Eq. (38) and (39) using the same spatial difference schemes, as was done for Eq. (19) and (22), shows that the new equations possess much smaller error amplification factors for the same δt (in particular, in the case of upwind spatial differencing, it is less than unity unconditionally). It can also be shown, as is done in the Appendix, that the formal order of accuracy of the splitting scheme remains unaltered by the introduction of Eq. (38) and (39) in place of Eq. (19) and (22). In practice, however, the accuracy of the new formulation (compared against the exact solution of the difference equations) was found to be significantly better, especially for large δt, than the initial formulation.

Final Formulation

Presently, the final form of the operator-split equations, including the modifications deemed necessary by the accuracy and stability consideration presented above, will be stated. Before doing so, however, one final consideration must be taken into account.

In practice, the velocity-divergence field $\Delta_i u_i^n$ may not vanish exactly at all times for the following reasons: either (i) the initial conditions do not satisfy continuity, or (ii) the pressure equation is not solved exactly at each time-step[4]. The term $\Delta_i u_i^n$ should therefore be retained in each of the pressure equations solved, otherwise mass errors may accumulate during the calculations. Alternatively, the operator-split equations can be re-cast into ones in terms of increments of variables rather than in terms of their absolute values. This eliminates the $\Delta_i u_i^n$ term, as well as other terms (such as the sources) from the equations, hence minimising the computing effort and storage requirements. The final equations are now as follows.

(a) *Predictor Step.* The momentum equation for u_i^* is carried over from Eq. (17). With the introduction of the H' operator in Eq. (37), it can be rewritten as

$$\left(\frac{\rho}{\delta t} - A_0\right) u_i^* = H'(u_i^*) - \Delta_i p^n + S_i + \frac{\rho}{\delta t} u_i^n. \tag{40}$$

This implicit equation is solved for u_i^*.

(b) *First Corrector Step.* The operative momentum equation (38) is transformed by subtracting Eq. (40) from it to

$$\left(\frac{\rho}{\delta t} - A_0\right)(u_i^{**} - u_i^*) = -\Delta_i(p^* - p^n) \tag{41}$$

which may be regarded as a velocity-increment equation. When the divergence of this equation is combined with the continuity relation (18), the following pressure-increment equation is obtained:

$$\Delta_i\left[\left(\frac{\rho}{\delta t} - A_0\right)^{-1} \Delta_i\right](p^* - p^n) = \Delta_i u_i^*. \tag{42}$$

Equation (42) is solved for the $(p^* - p^n)$ field, which is then inserted into Eq. (41) to get the new velocity field, u_i^{**}.

Second Corrector Step. Subtraction of Eq. (40) from the momentum equation (39) governing this step gives

$$\left(\frac{\rho}{\delta t} - A_0\right)(u_i^{***} - u_i^{**}) = H'(u_i^{**} - u_i^*) - \Delta_i(p^{**} - p^*). \tag{43}$$

This equation, combined with the continuity relation (21), yields the pressure-increment equation

$$\Delta_i\left[\left(\frac{\rho}{\delta t} - A_0\right)^{-1} \Delta_i\right](p^{**} - p^*) = \Delta_i\left[\left(\frac{\rho}{\delta t} - A_0\right)^{-1} H'(u_i^{**} - u_i^*)\right]. \tag{44}$$

[4] This occurs when the equation is solved by one of the standard iterative methods, in which case the solution is converged to only within some specified convergence criterion.

Equation (44) is solved for the $(p^{**} - p^*)$ field, which is then used in conjunction with Eq. (43) to obtain u_i^{***}. This completes the splitting process at which stage the u_i^{***} and p^{**} fields are taken to stand for the exact solution u_i^{n+1} and p^{n+1}.

At this stage, it is appropriate to compare the incompressible version of PISO just presented with other apparently similar, though iterative, schemes such as SIMPLER [8] and PUP [14]. Both of these were developed for steady-state calculations, although they may be applied to transient flows whereby iteration is used at each time-step. In particular, PUP can be shown to have a close resemblance to PISO, the main difference being the absence of a final velocity update in the former. Altough the differences may seem superficial, they are of significance. For, whereas the final velocity and pressure fields for a time-step in PISO, i.e., u_i^{***} and p^{**}, satisfy the one and same momentum equation (39), the same is not true for the fields obtained at the end of one iteration of either of the other two methods. It is, therefore, debatable whether these fields can be regarded as legitimate approximations to the solution of the difference equations over the time step (as in PISO).

Boundary Conditions

Two types of boundary conditions are encountered in practical fluid flow problems, these are prescribed velocity (or its gradient) and prescribed pressure boundaries. The splitting procedure handles both of these constraints with ease, especially when employed in conjunction with a staggered mesh arrangement as that used in, for example, [2, 5]. In what follows, the decoupling of the variables at the boundaries is discussed.

When the velocity at the boundary is prescribed (here only the velocity component normal to the boundary is of concern), all intermediate values of u_i at the boundary, namely, u_i^*, u_i^{**}, and u_i^{***}, are set to the given boundary value. From Eq. (41) and (43), it follows that $\Delta_i(p^* - p^n)$ and $\Delta_i(p^{**} - p^*)$ are zero at that boundary; these serve as boundary conditions for the pressure-increment equations (42) and (44). It should be noted, however, that the latter equations possess a solution only when the algebraic sum of the right-hand side of each of them over the whole field is zero. This condition corresponds to the satisfaction of overall continuity across all the boundaries of the domain, and this must be ensured at each of the splitting stages.

Alternatively, if the pressure at the boundary is specified, then p^* and p^{**} are set to the prescribed value, which serves as the boundary condition for Eq. (42) and (44). The normal velocities at the boundaries are updated in the usual manner using Eq. (41) and (43).

Generalisation to Compressible Flows

In what preceded, the method of splitting of operations was applied to the incompressible flow equations. Presently, the technique is to be generalised to encompass the compressible flow equations as well; this generalisation in principle is unrestricted to any particular Mach number regime. The equations for which a solution is sought now are Eq. (7), (9), and (13), the last of which embodies the continuity relation (8). The auxiliary equation of state (3) which couples together the pressure, density, and temperature (hence energy) must also be invoked. The splitting of these equations follows directly from the final formulations arrived at earlier for incompressible flow. However, because the coupling between the equations now involves the density and temperature, an additional corrector stage must be incorporated to achieve the same formal order of accuracy as with the linearised incompressible case (i.e., $O(\delta t^3)$ discretisation errors). This level of accuracy may often be higher than that of the temporal difference scheme used, in which case the use of such high-accuracy solution algorithm becomes unwarranted in view of the additional effort involved. Alternatively, a simpler but lower-order scheme involving only two corrector stages may be implemented. This latter scheme has proved to be of satisfactory accuracy as tests in [10] verify. Such a finding is in line with the earlier reasoning presented for the incompressible flow case that the merits of the present technique stem largely from its ability to resolve a pressure field free from the influence of errors in the divergence of the calculated velocities, a feature which is carried over to the compressible flow case.

In what follows, both the two and three corrector stage schemes are presented unaccompanied by an accuracy analysis of the kind presented earlier. Such an analysis has been carried out for the case of a perfect gas but proves to be too cumbersome for presentation here. The results of the analysis show that the two-stage method gives velocity, pressure, and temperature fields which differ from the exact solution by terms of $O(\delta t^2)$, while the three-stage scheme reduces these errors to $O(\delta t^3)$.

Two-Stage Scheme

(a) *Momentum Predictor Step.* The equation for momentum (7) is solved in this step implicitly, using old time pressures and densities, as

$$\left(\frac{1}{\delta t}-\frac{A_0}{\rho^n}\right)\rho^n u_i^* = H'(u_i^*) - \Delta_i p^n + S_i + \frac{\rho^n u_i^n}{\delta t}. \tag{45}$$

The solution of this equation yields u_i^*.

(b) *First Momentum Corrector Step.* The momentum equation is now written in the explicit corrector form

$$\left(\frac{1}{\delta t}-\frac{A_0}{\rho^n}\right)\rho^* u_i^{**} = H'(u_i^*) - \Delta_i p^* + S_i + \frac{\rho^n u_i^n}{\delta t} \tag{46}$$

which, by subtracting Eq. (45) from it, can be re-cast in incremental form as

$$\rho^* u_i^{**} - \rho^n u_i^* = -\left(\frac{1}{\delta t} - \frac{A_0}{\rho^n}\right)^{-1} \Delta_i(p^* - p^n). \tag{47}$$

The continuity equation (8) is now taken as

$$\Delta_i(\rho^* u_i^{**}) = -\frac{1}{\delta t}(\rho^* - \rho^n). \tag{48}$$

Taking the divergence of Eq. (47) and invoking Eq. (48) gives

$$\Delta_i\left[\left(\frac{1}{\delta t} - \frac{A_0}{\rho^n}\right)^{-1} \Delta_i\right](p^* - p^n) = \Delta_i(\rho^n u_i^*) + \frac{1}{\delta t}(\rho^* - \rho^n) \tag{49}$$

where ρ^* must be eliminated in favour of p^* to enable the solution of the equation. This is done by writing the equation of state (3) as

It is more convenient, however, to use the incremental form (obtained by subtraction of Eq. (52) from (58))

$$\left(\frac{1}{\delta t}-\frac{B_0}{\rho^{**}}\right)\rho^{**}e^{**}-\left(\frac{1}{\delta t}-\frac{B_0}{\rho^{*}}\right)\rho^{*}e^{*}=-\Delta_i[u_i^{***}p^{**}-u_i^{**}p^{*}]. \tag{59}$$

The temperature T^{**} is evaluated from e^{**} and u_i^{***}.

(f) *Third Momentum Corrector Step.* In this final step, the momentum equation is now written as

$$\left(\frac{1}{\delta t}-\frac{A_0}{\rho^{**}}\right)\rho^{***}u_i^{\overset{**}{**}}=H'(u_i^{**})-\Delta_i p^{***}+S_i+\frac{\rho^n u_i^n}{\delta t} \tag{60}$$

which, in incremental form, becomes

$$\rho^{***}u_i^{\overset{**}{**}}-\rho^{**}u_i^{***}=\left(\frac{1}{\delta t}-\frac{A_0}{\rho^{**}}\right)^{-1}\left\{-\Delta_i(p^{***}-p^{**})\right.$$
$$\left.-A_0\left(\frac{\rho^{**}-\rho^n}{\rho^n}\right)u_i^{***}\right\}. \tag{61}$$

The continuity relation is now written as

$$\Delta_i(\rho^{***}u_i^{\overset{**}{**}})=-\frac{1}{\delta t}(\rho^{***}-\rho^n). \tag{62}$$

Combination of Eq. (61) and (62) gives the pressure equation

$$\left[\Delta_i\left\{\left(\frac{1}{\delta t}-\frac{A_0}{\rho^{**}}\right)^{-1}\Delta_i\right\}-\frac{\phi(p^{**},T^{**})}{\delta t}\right](p^{***}-p^{**})$$
$$\Delta_i\left[-\left(\frac{1}{\delta t}-\frac{A_0}{\rho^{**}}\right)^{-1}A_0\left(\frac{\rho^{**}-\rho^n}{\rho^n}\right)u_i^{***}\right]$$
$$+\frac{p^{**}}{\delta t}[\phi(p^{**},T^{**})-\phi(p^{*},T^{*})] \tag{63}$$

where the equation of state

$$\rho^{***}=p^{***}\phi(p^{**},T^{**}) \tag{64}$$

has been used in arriving at Eq. (63).

Equation (63) now yields p^{***}, while Eq. (64) and (61) give ρ^{***} and $u_i^{\overset{**}{**}}$ respectively. This completes the splitting whereby the latest computed fields are now regarded as the final solutions to the original equations (7), and (8), (9), and (3).

Here, the resemblance between PISO and the iterative SIMPLER [8] and PUP

[14] becomes much more remote than is the case for incompressible flow. The non-iterative strategy of PISO adjusts the mass-velocity as a whole at each stage to satisfy continuity and momentum simultaneously. In contrast, the other methods either update the density at the end of the iteration or introduce a density correction (but only in the continuity equation), both practices of which demand iteration.

Generalisation of the Method to Other Equations

The splitting procedure has so far been applied to the equations of motion only. It is often the case that other scalar transport equations have to be solved in conjunction with those for motion, e.g., equations for the turbulence kinetic energy, k, and its rate of dissipation, ε, which arise in the well-established $k-\varepsilon$ model of turbulence (see [11]). These equations are usually coupled strongly together, especially through the source terms. As the splitting method presented earlier does away with iteration, a non-iterative scheme must also be developed to deal with these other equations such that the accuracy and stability of the overall scheme are preserved. Indeed, it is often the case that it is the poor resolution of these scalar fields that undermines the integrity of the overall solution procedure.

In this section, a generalisation of the splitting technique to the solution of implicitly discretised scalar equations that are coupled through source terms is proposed. The method is illustrated by application to the k and ε equations of the aforementioned turbulence model. These equations in differential form are

$$\frac{\partial}{\partial t}(\rho k)+\frac{\partial}{\partial x_j}(\rho u_j k)=\frac{\partial}{\partial x_j}\left(\Gamma\frac{\partial k}{\partial x_j}\right)+S_k \tag{65}$$

and

$$\frac{\partial}{\partial t}(\rho\varepsilon)+\frac{\partial}{\partial x_j}(\rho u_j\varepsilon)=\frac{\partial}{\partial x_j}\left(\Gamma\frac{\partial}{\partial x_j}\varepsilon\right)+S_\varepsilon \tag{66}$$

where Γ is a diffusion coefficient for the quantities k and ε. The source terms S_k and S_ε couple the above equations strongly, and take the form

$$S_k \equiv \mu g - \rho\varepsilon \tag{67}$$

$$S_\varepsilon \equiv C_1 g\mu\varepsilon/k - C_2\rho\varepsilon^2/k \tag{68}$$

where g is a generation term (related to the velocity strain field), and μ is the turbulent viscosity which is related to k and ε via the relation

$$\mu = C_\mu\frac{\rho k^2}{\varepsilon}. \tag{69}$$

The quantities C_1, C_2, and C_μ are empirical coefficients in the turbulence model.

When the above transport equations (65) and (66) are implicitly discretised (here again the Euler implicit scheme is used for convenience), the resulting finite-difference equations take the form

$$\frac{1}{\delta t}\{(\rho k)^{n+1}-(\rho k)^{n}\}=K(k^{n+1})+(\mu g)^{n+1}-(\rho\varepsilon)^{n+1} \tag{70}$$

and

$$\frac{1}{\delta t}\{(\rho\varepsilon)^{n+1}-(\rho\varepsilon)^{n}\}=L(\varepsilon^{n+1})+(C_1\mu g\varepsilon/k)^{n+1}-(C_2\rho\varepsilon^2/k)^{n+1} \tag{71}$$

where the operators K and L represent the discrete analogues of the convection and diffusion terms in the parent equations. The task now is to develop a splitting scheme which will enable the non-iterative solution of these equations, given the coupling between them via the sources S_k and S_ε.

For simplicity of presentation, it will be assumed that the solution of k and ε will not affect the equations of motion. This would be the case if the latter equations are solved based on the old time level turbulent viscosity. Hence, the quantities g^{n+1} and ρ^{n+1} in Eq. (70) and (71) will be treated as known and are determined by the solution of the momentum, continuity, and energy equations. Equations (70) and (71) are now rearranged, first by taking the central elements from the K and L operators and shifting them to the left-hand side of the equations (in keeping with the practice adopted for the equations of motion). Second, by invoking relation (69), the source terms S_k and S_ε in expressions (67) and (68) are transformed into the forms

$$S_k=\mu g-\frac{\rho^2 C_\mu}{\mu}k^2 \tag{72}$$

and

$$S_\varepsilon=C_1 C_\mu g\rho k-C_2 C_\mu\rho^2\frac{k}{\mu}\varepsilon \tag{73}$$

which are more suitable for the splitting to be introduced than their original counterparts. With those changes introduced, Eq. (70) and (71) now become

$$\left(\frac{\rho}{\delta t}-C_0+\frac{\rho^2 C_\mu}{\mu^{n+1}}k^{n+1}\right)k^{n+1}=K'(k^{n+1})+\mu^{n+1}g+\frac{\rho^n k^n}{\delta t} \tag{74}$$

and

$$\left(\frac{\rho}{\delta t}-D_0+C_2 C_\mu\rho^2\frac{k^{n+1}}{\mu^{n+1}}\right)\varepsilon^{n+1}=L'(\varepsilon^{n+1})+C_1 g C_\mu\rho k^{n+1}+\frac{\rho^n\varepsilon^n}{\delta t} \tag{75}$$

where K' and L' are the spatial flux operators minus their central elements $C_0 k$ and $D_0 \varepsilon$, respectively, which now appear on the left-hand side. Also shifted to the left side of the equations are the negative contributions of the source terms; this is to ensure that negative values for k and ε (which is a non-physical solution) are never generated. Note that the unsuperscripted quantities in Eq. (74) and (75) are known and pertain to time level $n+1$. The solution of these equations by operator-splitting can now be achieved as follows:

Predictor Step. The above equations are effectively decoupled by writing them as

$$\left(\frac{\rho}{\delta t} - C_0 + \frac{\rho^2 C_\mu}{\mu^n} k^n\right) k^* = K'(k^*) + \mu^n g + \frac{\rho^n k^n}{\delta t} \tag{76}$$

and

$$\left(\frac{\rho}{\delta t} - D_0 + C_2 C_\mu \rho^2 \frac{k^*}{\mu^n}\right) \varepsilon^* = L'(\varepsilon^*) + C_1 g C_\mu \rho k^* + \frac{\rho^n \varepsilon^n}{\delta t}. \tag{77}$$

Equations (76) and (77) are implicit in k^* and ε^*, respectively, and are to be solved in that sequence.

The viscosity μ^* can now be calculated from

$$\mu^* = C_\mu \rho \frac{k^{*2}}{\varepsilon^*}. \tag{78}$$

Corrector Step. The corrector equations for k and ε are explicit-like and take the forms

$$\left(\frac{\rho}{\delta t} - C_0 + \frac{\rho^2 C_\mu}{\mu^*} k^*\right) k^{**} = K'(k^*) + \mu^* g + \frac{\rho^n k^n}{\delta t} \tag{79}$$

and

$$\left(\frac{\rho}{\delta t} - D_0 + C_2 C_\mu \rho^2 \frac{k^{**}}{\mu^*}\right) \varepsilon^{**} = L'(\varepsilon^*) + C_1 g C_\mu \rho k^{**} + \frac{\rho^n \varepsilon^n}{\delta t}. \tag{80}$$

It is more convenient to use an incremental form of Eq. (79) and (80) by subtracting Eq. (76) and (77) from them. Thus

$$k^{**} = \left(\frac{\rho}{\delta t} - C_0 + \frac{\rho^2 C_\mu k^*}{\mu^*}\right)^{-1} \left[\left(\frac{\rho}{\delta t} - C_0 + \frac{\rho^2 C_\mu k^n}{\mu^n}\right) k^* + (\mu^* - \mu^n) g\right] \tag{81}$$

and

$$\varepsilon^{**} = \left(\frac{\rho}{\delta t} - D_0 + C_2 C_\mu \rho^2 \frac{k^{**}}{\mu^*}\right)^{-1} \left[\left(\frac{\rho}{\delta t} - D_0 + C_2 C_\mu \rho^2 \frac{k^*}{\mu^n}\right) \varepsilon^* + C_1 \rho C_\mu g (k^{**} - k^*)\right]. \tag{82}$$

Equations (81) and (82) yield the new values of k^{**} and ε^{**}. The new viscosity is now given by

$$\mu^{**} = C_\mu \rho \frac{k^{**2}}{\varepsilon^{**}}. \tag{83}$$

It can be shown that k^{**} and ε^{**} are second-order approximations (in δt) to k^{n+1} and ε^{n+1}. It is possible, though often unnecessary, to introduce further corrector stages along the same lines outlined above in order to increase the accuracy. This, however, is not presented here, as the derivation follows closely that given above.

Conclusions

In what preceded, a non-iterative method of handling the pressure/velocity coupling arising in the implicitly discretised fluid flow equations is presented. This is accomplished by the splitting of the process of solution into a series of predictor and corrector steps such that, in each step, simplified equations result, whose solutions can be achieved by existing standard techniques. The fields obtained at the end of these steps are approximations to the exact ones with a temporal accuracy comparable to, or better than, the accuracy of the discretisation scheme used for deriving the finite-difference analogue of the original differential equations.

The method is outlined initially by application to the incompressible flow equations, and, although the presentation was confined to equations discretised using the Euler implicit scheme, the method can, in principle, be generalised to include other temporal differencing schemes. The accuracy of the solution is analysed for this case and it is shown that the errors in the solution of the linearised difference equations for pressure and velocity are of the order δt^3 and δt^4, respectively.

The method is then extended to the implicitly discretised compressible flow equations, albeit also discretised by the Euler implicit scheme. Two versions of the method, one with two corrector steps and the other with three, are presented. The accuracy of these versions (with a perfect gas assumption) has been analysed (although not presented) and has been found to be of the order of δt^2 and δt^3 (in discretisation errors), respectively.

The method has been applied to the calculation of transient flows in both compressible and incompressible situations. The results of these calculations are presented in [10], where it is amply demonstrated that the method is:

(i) of good temporal accuracy;

(ii) faster than iterative techniques for transient flow;

(iii) stable for large time-steps (hence making it efficient for steady-state problems as well as transient ones);

(iv) applicable to incompressible as well as compressible flow régimes, Finally, a generalisation of the technique to deal with other scalar equations is proposed. The algorithm is illustrated by application to the source-coupled equations of the $k-\varepsilon$ turbulence model. This development has been partially tested with much success and the work will be reported upon completion.

APPENDIX

Accuracy Considerations of the Final Form of the Splitting Scheme

In the text, the temporal accuracy of the initial formulation of PISO is assessed. That formulation is modified upon further consideration of the stability of the splitting procedure, thus leading to Eq. (38) and (39) in place of (19) and (20). The task is to examine the consequences of the modification (namely, the implicit treatment of the central element in the finite-difference expressions for the convective and diffusive fluxes in these equations) on the accuracy and consistency of the scheme.

The predictor-stage equation is unaltered and the argument presented in the text is valid here also. Hence, the error in the u_i^* field is

$$\varepsilon_i^* = O(\delta t^2).$$

The new forms of the first corrector error equations (corresponding to Eq. (27) and (28) in the first formulation) are now

$$\left(\frac{\rho}{\delta t} - A_0\right)\varepsilon_i^{**} = H'(\varepsilon_i^*) - \Delta_i \xi^* \tag{A.1}$$

and

$$\Delta_i\left[\left(\frac{\rho}{\delta t} - A_0\right)^{-1} \Delta_i\right]\xi^* = \Delta_i\left[\left(\frac{\rho}{\delta t} - A_0\right)^{-1} H'(\varepsilon_i^*)\right]. \tag{A.2}$$

In Eq. (A.2), it is clear that ξ^* vanishes with δt at the same rate as ε_i^*, i.e., ξ^* vanishes as δt^2. As a result, the right-hand side of Eq. (A.1) vanishes as δt^2. This implies that $(\rho/\delta t - A_0)\,\varepsilon_i^{**}$ must also vanish as δt^2, a condition which can only be satisfied if ε_i^{**} vanishes as δt^3. Similar arguments lead to the conclusion that ξ^{**} is $O(\delta t^3)$ and ε_i^{***} is $O(\delta t^4)$. This shows that the temporal accuracy of the scheme is unaffected by the introduction of Eq. (38) and (39).

So far, attention has been confined to the temporal accuracy of the splitting scheme with the implicit assumption that the spatial mesh size is kept finite. Unlike the initial formulation of the scheme, however, the final scheme raises the question as to whether an inconsistency is introduced by the implicit treatment of the central element $A_0 u_i$ of the $H(u_i)$ operator in the corrector stages. An analysis of this problem is very difficult in the context of the scheme, as the intermediate solutions

u_i^*, u_i^{**}, etc., do not relate to a precise time level, but are approximations to the exact solution u_i^{n+1} of the finite-difference equations, which themselves are approximations to the parent differential equations. Only a heuristic analysis of this question can therefore be offered here.

First, it is taken that the temporal and spatial discretisation schemes used in arriving at the finite difference equation (14) result in consistency with the differential equation (5). Next, it is assumed that the exact solution to Eq. (5) is U_i and P. Now, the final fields obtained from the splitting procedure are u_i^{***} and p^{**} which obey Eq. (39). The truncation error in this equation is therefore the remainder in the equation when U_i^{n+1} and P^{n+1} are substituted for u_i^{***} and p^{***} (see [13]). A problem arises here in that u_i^{**}, which also appears in the equation, cannot be related to the exact solution U_i^{n+1}, though it is supposed to be an approximation of it.

Equation (39) is first manipulated into the form

$$\frac{\rho}{\delta t} u_i^{***} - \frac{\rho}{\delta t} u_i^n = H(u_i^{***}) - H'(\varepsilon_i^{***} - \varepsilon_i^{**}) - \Delta_i p^{**} + S_i \tag{A.3}$$

whereby use of Eq. (37) has been made; also, the definition of ε_i in Eq. (24) is invoked. Comparison of Eq. (A.3) with Eq. (14) reveals that the term $H'(\varepsilon_i^{***} - \varepsilon_i^{**})$ is an intruder into (A.3) and can be the cause of inconsistency. Indeed, substitution of the exact solution U_i^{n+1}, P^{n+1} into the equations, and taking δt and δx_i (the spatial mesh size) to zero, gives the truncation error R as

$$R = H'(\varepsilon_i^{***} - \varepsilon_i^{**}), \qquad \delta t, \delta x_i \to 0 \tag{A.4}$$

where the fact that the discretised equation (14) is consistent with the differential equation (5) has been used. Consistency demands that R should vanish, and this depends on the nature of the spatial difference scheme which defines H' and on how ε_i vanishes as the mesh is refined.

In the absence of precise knowledge about ε_i^{**} and ε_i^{***}, a supposition is now introduced. It is that, as inferred earlier, ε_i^{**} and ε_i^{***} vanish as δt^3 and δt^4, respectively, indepently of δx_i. Once this is accepted, the truncation error in (A.4) can be shown to reduce to

$$R = A_0(\varepsilon_i^{***} - \varepsilon_i^{**}), \qquad \delta t, \delta x_i \to 0 \tag{A.5}$$

where definition (37) has been invoked. It can be inferred from expression (A.5) that R is dominated by ε_i^{**} (which is of lower order than ε_i^{***}) and should therefore vanish as $A_0 \varepsilon_i^{**}$. Since ε_i^{**} is $O(\delta t^3)$, then R vanishes as $A_0\, \delta t^3$. Now, A_0 is a coefficient in the spatial difference analogue of the convective and diffusive fluxes; it contains terms in $1/\delta x_i$. For consistency, therefore, A_0^{-1} must not vanish as fast or faster than δt^3.

Typically for the upwind scheme, for example, $A_0 = -[(\rho u_i/\delta x_i) + (2\mu/\delta x_i^2)]$. Hence, R vanishes as $\delta t^3/\delta x_i$. For a centred scheme, R vanishes as $\delta t^3/\delta x_i^2$. In both cases, the truncation error vanishes at a faster rate with δt than with δx_i. This is useful, since in practical computations δt is usually taken (in the worst event) as proportional to δx_i in which case R goes to zero and the above schemes are consistent with the differential equations.

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